

FUNCTIONAL RELATIONS FOR ZETA-FUNCTIONS OF WEIGHT LATTICES OF LIE GROUPS OF TYPE A_3

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ABSTRACT. We study zeta-functions of weight lattices of compact connected semisimple Lie groups of type A_3 . Actually we consider zeta-functions of $SU(4)$, $SO(6)$ and $PU(4)$, and give some functional relations and new classes of evaluation formulas for them.

1. INTRODUCTION

For any semisimple Lie algebra \mathfrak{g} , the Witten zeta-function $\zeta_W(s; \mathfrak{g})$ is defined by

$$(1.1) \quad \zeta_W(s; \mathfrak{g}) = \sum_{\rho} (\dim \rho)^{-s},$$

where $s \in \mathbb{C}$ and ρ runs over all finite dimensional irreducible representations of \mathfrak{g} . This was formulated by Zagier (see [26, Section 7]), who was inspired by Witten's work [25]. Witten's motivation of introducing the above zeta-functions is to express the volumes of certain moduli spaces in terms of special values of $\zeta_W(s; \mathfrak{g})$. The result is called Witten's volume formula, and from which it can be shown that

$$(1.2) \quad \zeta_W(2k; \mathfrak{g}) = C_W(2k, \mathfrak{g}) \pi^{2kn}$$

for any $k \in \mathbb{N}$, where $C_W(2k, \mathfrak{g}) \in \mathbb{Q}$ (see [26, Theorem, p.506]). The explicit value of $C_W(2k, \mathfrak{g})$ was not determined in their work.

Gunnells and Sczech introduced a method to compute $C_W(2k, \mathfrak{g})$ explicitly (see [3]). The theory of Szenes ([21], [22]) also gives a different algorithm of computing $C_W(2k, \mathfrak{g})$.

Let r be the rank of \mathfrak{g} , $\Delta(\mathfrak{g})$ the root system corresponding to \mathfrak{g} , and n the number of positive roots belonging to $\Delta(\mathfrak{g})$. In [6, 7, 11, 19], the authors defined the zeta-function $\zeta_r(\mathbf{s}; \Delta(\mathfrak{g}))$ of the root system $\Delta(\mathfrak{g})$, where $\mathbf{s} = (s_i) \in \mathbb{C}^n$ (see Section 3). This may be regarded as a multi-variable version of $\zeta_W(s; \mathfrak{g})$ (see also survey papers [10], [16]). The authors further introduced a generalization of Bernoulli polynomials associated with root systems in [7, 9, 13]. Using these tools, we can generalize (1.2) ([13, Theorem 4.6]) and express $C_W(2k, \mathfrak{g})$ in terms of Bernoulli polynomials of root systems (see also [10]), hence gives another algorithm for computing $C_W(2k, \mathfrak{g})$.

Moreover we can give various functional relations for zeta-functions of root systems (see [6, 7, 8, 9, 13, 14, 19]; we will discuss this matter further in the next section).

More recently, the authors defined zeta-functions of weight lattices of compact connected semisimple Lie groups (see [17]). If the group is simply-connected, these zeta-functions coincide with ordinary zeta-functions of root systems of associated Lie algebras. We considered the general connected (but not necessarily simply-connected) case and proved a result analogous to (1.2) for these zeta-functions, and further prove functional relations among them. The present paper is a continuation of [17], and we study zeta-functions of lattices of Lie groups whose associated Lie algebras are of type A_3 . The reason why we treat the case A_3 will be mentioned in Section 3.

Throughout this paper, let \mathbb{N} be the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, \mathbb{Z} the ring of rational integers, \mathbb{Q} the rational number field, \mathbb{R} the real number field, and \mathbb{C} the complex number field.

2. FUNCTIONAL RELATIONS: A MOTIVATION

In this section we comment our motivation on the study of functional relations.¹ The Euler-Zagier r -ple sum is defined by

$$(2.1) \quad \zeta_{EZ,r}(\mathbf{s}) = \sum_{m_1, \dots, m_r=1}^{\infty} \frac{1}{m_1^{s_1} (m_1 + m_2)^{s_2} \cdots (m_1 + \cdots + m_r)^{s_r}},$$

where $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$ (see [4], [26]). Special values of (2.1) for $\mathbf{s} = \mathbf{k}$, where $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$ with $k_r \geq 2$, are known to be important in various fields of mathematics. Euler already obtained the following relations among those values in the case $r = 2$: The harmonic product relation

$$(2.2) \quad \zeta(k_1)\zeta(k_2) = \zeta_{EZ,2}(k_1, k_2) + \zeta_{EZ,2}(k_2, k_1) + \zeta(k_1 + k_2),$$

where $\zeta(s) = \zeta_{EZ,1}(s)$ is the Riemann zeta-function, and the sum formula

$$(2.3) \quad \sum_{j=2}^{k-1} \zeta_{EZ,2}(k-j, j) = \zeta(k)$$

for $k \in \mathbb{N}$, $k \geq 3$. After the discovery of the importance of Euler-Zagier sums (around 1990, according to the work of Drinfel'd, Goncharov, Kontsevich, Hoffman and Zagier), many people began to search for various relations among special values of (2.1), and indeed a lot of relations have been discovered.

¹The contents of this section was given in the talk of the second-named author on the problem session of the conference.

Around 2000, the second-named author raised a question: are those relations valid only at positive integers, or valid also continuously at other values?

In fact, it is easy to see that (2.2) is valid for any complex numbers s_1, s_2 except for singularities, that is

$$(2.4) \quad \zeta(s_1)\zeta(s_2) = \zeta_{EZ,2}(s_1, s_2) + \zeta_{EZ,2}(s_2, s_1) + \zeta(s_1 + s_2).$$

Therefore the harmonic product relation is actually a “functional relation”. So far, except for (2.4) and its relatives, no other such functional relations among Euler-Zagier sums has been discovered. (In the double zeta case, the functional equation [12] is known, but it is not a formula which interpolates some known value-relation.)

However, when we consider more extended classes of multiple series, we can find a lot of functional relations! The first examples were reported by the third-named author [23], in which functional relations among $\zeta(s)$ and the Tornheim double sum

$$(2.5) \quad \zeta_{MT,2}(s_1, s_2, s_3) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}}$$

are proved. Those relations can be regarded as functional relations among zeta-functions of root systems, because $\zeta_{MT,2}(s_1, s_2, s_3)$ coincides with the zeta-function of the root system $\Delta(\mathfrak{su}(3))$.

It is known that irreducible root systems are classified as types X_r (where X is one of A, B, C, D, E, F, G) by the Killing-Cartan theory. When $\Delta(\mathfrak{g})$ is of type X_r , we denote the corresponding zeta-function as $\zeta_r(\mathbf{s}; X_r)$. Using this notation, we see that (2.5) is $\zeta_2(\mathbf{s}; A_2)$.

Various other functional relations have then been proved in several articles of the authors: on A_2 ([13]), on A_3 ([19], [6], [8], [15]), on $B_2 = C_2$ ([8], [9]), on B_3 and C_3 ([8]), and on G_2 ([14]). A general treatment on the theory of functional relations is given in [13], [9]. Those functional relations in fact include various known value-relations among special values of Euler-Zagier sums, and also include Witten’s formula (1.2) in several cases.

Moreover, functional relations also exist among zeta-functions of lattices of (not necessarily simply-connected) Lie groups in the sense of [17]. In [17], we proved functional relations among zeta-functions whose associated root systems are of type A_2 or C_2 .

In this paper we study zeta-functions of weight lattices of compact connected semisimple Lie groups of type A_3 . More precisely, we consider zeta-functions of $SU(4)$, $SO(6)$ and $PU(4)$.

In Section 3, we recall the definition of those zeta-functions. In Section 4, we prepare some lemmas which will be necessary later. Then in the remaining sections

we prove some functional relations for those zeta-functions which are the main results in this paper (see Theorems 8, 9, 12, 17 and 18). Moreover we give new classes of evaluation formulas for these zeta-functions in terms of the Riemann zeta-function (see Propositions 10 and 13, and Examples 11 and 14) and the Dirichlet L -function with the primitive character of conductor 4 (see Proposition 19 and Examples 20 and 21).

3. ZETA-FUNCTIONS OF WEIGHT LATTICES

In this section, we recall the definition and some properties of zeta-functions of weight lattices of compact connected semisimple Lie groups which we considered in our previous paper [17, Section 3].

We first prepare the same notation as in [9, 11, 13] (see also [6, 7, 10, 14]). Let V be an r -dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. The norm $\|\cdot\|$ is defined by $\|v\| = \langle v, v \rangle^{1/2}$. The dual space V^* is identified with V via the inner product of V . Let Δ be a finite reduced root system which may not be irreducible, and $\Psi = \{\alpha_1, \dots, \alpha_r\}$ its fundamental system. We fix Δ_+ and Δ_- as the set of all positive roots and negative roots respectively. Then we have a decomposition of the root system $\Delta = \Delta_+ \amalg \Delta_-$. Let $Q = Q(\Delta)$ be the root lattice, Q^\vee the coroot lattice, $P = P(\Delta)$ the weight lattice, P^\vee the coweight lattice, and P_+ the set of integral dominant weights defined by

$$(3.1) \quad Q = \bigoplus_{i=1}^r \mathbb{Z} \alpha_i, \quad Q^\vee = \bigoplus_{i=1}^r \mathbb{Z} \alpha_i^\vee,$$

$$(3.2) \quad P = \bigoplus_{i=1}^r \mathbb{Z} \lambda_i, \quad P^\vee = \bigoplus_{i=1}^r \mathbb{Z} \lambda_i^\vee,$$

$$(3.3) \quad P_+ = \bigoplus_{i=1}^r \mathbb{N}_0 \lambda_i,$$

respectively, where the fundamental weights $\{\lambda_j\}_{j=1}^r$ and the fundamental coweights $\{\lambda_j^\vee\}_{j=1}^r$ are the dual bases of Ψ^\vee and Ψ satisfying $\langle \alpha_i^\vee, \lambda_j \rangle = \delta_{ij}$ and $\langle \lambda_i^\vee, \alpha_j \rangle = \delta_{ij}$ respectively. Let

$$(3.4) \quad \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \sum_{j=1}^r \lambda_j$$

be the lowest strongly dominant weight. Let σ_α be the reflection with respect to a root $\alpha \in \Delta$ defined as

$$(3.5) \quad \sigma_\alpha : V \rightarrow V, \quad \sigma_\alpha : v \mapsto v - \langle \alpha^\vee, v \rangle \alpha.$$

For a subset $A \subset \Delta$, let $W(A)$ be the group generated by reflections σ_α for all $\alpha \in A$. In particular, $W = W(\Delta)$ is the Weyl group, and $\{\sigma_j = \sigma_{\alpha_j} \mid 1 \leq j \leq r\}$ generates W .

Let \tilde{G} be a simply-connected compact connected semisimple Lie group, and $\mathfrak{g} = \text{Lie}(\tilde{G})$. There is a one-to-one correspondence between a compact connected semisimple Lie group G whose universal covering group is \tilde{G} , and a lattice L with $Q(\Delta(\mathfrak{g})) \subset L \subset P(\Delta(\mathfrak{g}))$ up to automorphisms by taking $L = L(G)$ as the weight lattice of G . Let $L_+ = P_+ \cap L$.

Now we recall the definition of the zeta-function of the weight lattice $L = L(G)$ of the semisimple Lie group G , that is,

$$(3.6) \quad \zeta_r(\mathbf{s}, \mathbf{y}; G) = \zeta_r(\mathbf{s}, \mathbf{y}; L; \Delta) := \sum_{\lambda \in L_+ + \rho} e^{2\pi i \langle \mathbf{y}, \lambda \rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}},$$

where $\mathbf{y} \in V$. Note that this zeta-function can be continued meromorphically to \mathbb{C}^n . When $\mathbf{y} = \mathbf{0}$, we sometimes write this zeta-function as $\zeta_r(\mathbf{s}; G)$ or $\zeta_r(\mathbf{s}; L; \Delta)$ for brevity. It is to be noted that if $G = \tilde{G}$, then $L = P$ and $\zeta_r(\mathbf{s}; \tilde{G})$ coincides with $\zeta_r(\mathbf{s}; \mathfrak{g})$, also written as $\zeta_r(\mathbf{s}; \Delta(\mathfrak{g}))$ and $\zeta_r(\mathbf{s}; X_r)$ when $\Delta(\mathfrak{g})$ is of type X_r , which is called the zeta-function of the root system of type X_r studied in our previous papers (see, for example, [6, 10, 11, 13]).

In the present paper we concentrate our attention on the case of type A_3 , so we write down the explicit form of zeta-functions in this case. Let $\Delta = \Delta(A_3)$ with $\Psi = \{\alpha_1, \alpha_2, \alpha_3\}$, $\Delta_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$, $P = \sum_{j=1}^3 \mathbb{Z}\lambda_j$ and $Q = \sum_{j=1}^3 \mathbb{Z}\alpha_j$. It is known that $P/Q \simeq \mathbb{Z}/4\mathbb{Z}$. Therefore there is a unique intermediate lattice L_1 with $P \supsetneq L_1 \supsetneq Q$, satisfying $(L_1 : Q) = 2$. The group corresponding to P (resp. Q) is $SU(4)$ (resp. $PU(4)$). The group $G = G(L_1)$ is $SU(4)/\{\pm 1\}$, which is known to be isomorphic to $SO(6)$. We know (for the details, see [17, Example 4.3]) that

$$(3.7) \quad \begin{aligned} \zeta_3(\mathbf{s}, \mathbf{y}; SU(4)) &= \zeta_3(\mathbf{s}, \mathbf{y}; P; A_3) \\ &= \sum_{m_1, m_2, m_3=1}^{\infty} \frac{e^{2\pi i \langle \mathbf{y}, m_1 \lambda_1 + m_2 \lambda_2 + m_3 \lambda_3 \rangle}}{m_1^{s_1} m_2^{s_2} m_3^{s_3} (m_1 + m_2)^{s_4} (m_2 + m_3)^{s_5} (m_1 + m_2 + m_3)^{s_6}}, \end{aligned}$$

with

$$\lambda_1 = \frac{3}{4}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{4}\alpha_3, \quad \lambda_2 = \frac{1}{2}\alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3, \quad \lambda_3 = \frac{1}{4}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{3}{4}\alpha_3.$$

Note that $\zeta_3(\mathbf{s}, \mathbf{0}; SU(4)) = \zeta_3(\mathbf{s}; A_3)$. Further we have

$$(3.8)$$

$$\zeta_3(\mathbf{s}, \mathbf{y}; SO(6)) = \zeta_3(\mathbf{s}, \mathbf{y}; L_1; A_3)$$

$$\begin{aligned}
&= \sum_{\substack{m_1, m_2, m_3=1 \\ m_1 \equiv m_3 \pmod{2}}}^{\infty} \frac{e^{2\pi i \langle \mathbf{y}, m_1 \lambda_1 + m_2 \lambda_2 + m_3 \lambda_3 \rangle}}{m_1^{s_1} m_2^{s_2} m_3^{s_3} (m_1 + m_2)^{s_4} (m_2 + m_3)^{s_5} (m_1 + m_2 + m_3)^{s_6}}, \\
(3.9) \quad &\zeta_3(\mathbf{s}, \mathbf{y}; PU(4)) = \zeta_3(\mathbf{s}, \mathbf{y}; Q; A_3) \\
&= \sum_{\substack{m_1, m_2, m_3=1 \\ m_1 + 2m_2 + 3m_3 \equiv 2 \pmod{4}}}^{\infty} \frac{e^{2\pi i \langle \mathbf{y}, m_1 \lambda_1 + m_2 \lambda_2 + m_3 \lambda_3 \rangle}}{m_1^{s_1} m_2^{s_2} m_3^{s_3} (m_1 + m_2)^{s_4} (m_2 + m_3)^{s_5} (m_1 + m_2 + m_3)^{s_6}}.
\end{aligned}$$

In particular,

$$\begin{aligned}
(3.10) \quad &\zeta_3(\mathbf{s}, \lambda_1^\vee; SU(4)) = \zeta_3(\mathbf{s}, \lambda_1^\vee; P; A_3) \\
&= \sum_{l, m, n=1}^{\infty} \frac{i^{3l+2m+n}}{l^{s_1} m^{s_2} n^{s_3} (l+m)^{s_4} (m+n)^{s_5} (l+m+n)^{s_6}},
\end{aligned}$$

$$\begin{aligned}
(3.11) \quad &\zeta_3(\mathbf{s}, \lambda_2^\vee; SU(4)) = \zeta_3(\mathbf{s}, \lambda_2^\vee; P; A_3) \\
&= \sum_{l, m, n=1}^{\infty} \frac{(-1)^{l+n}}{l^{s_1} m^{s_2} n^{s_3} (l+m)^{s_4} (m+n)^{s_5} (l+m+n)^{s_6}},
\end{aligned}$$

$$\begin{aligned}
(3.12) \quad &\zeta_3(\mathbf{s}, \lambda_3^\vee; SU(4)) = \zeta_3(\mathbf{s}, \lambda_3^\vee; P; A_3) \\
&= \sum_{l, m, n=1}^{\infty} \frac{i^{l+2m+3n}}{l^{s_1} m^{s_2} n^{s_3} (l+m)^{s_4} (m+n)^{s_5} (l+m+n)^{s_6}}.
\end{aligned}$$

Note that

$$(3.13) \quad \zeta_3((s_1, s_2, s_3, s_4, s_5, s_6), \lambda_1^\vee; SU(4)) = \zeta_3((s_3, s_2, s_1, s_5, s_4, s_6), \lambda_3^\vee; SU(4)),$$

$$(3.14) \quad \zeta_3((s_1, s_2, s_3, s_4, s_5, s_6), \lambda_2^\vee; SU(4)) = \zeta_3((s_3, s_2, s_1, s_5, s_4, s_6), \lambda_2^\vee; SU(4)),$$

as well as

$$(3.15) \quad \zeta_3((s_1, s_2, s_3, s_4, s_5, s_6); A_3) = \zeta_3((s_3, s_2, s_1, s_5, s_4, s_6); A_3).$$

There are several reasons why the study on the case A_3 deserves one paper. First, since the case A_2 was studied in [17], it is a natural continuation. Second, since the functional relation for $\zeta_3(\mathbf{s}; A_3)$ given in [8] is restricted to the case of even integers, we supply a more general result (Theorem 9) here. The third reason is that Lie algebras of type A_r are the most interesting in view of the theory of [17], because $|P(\Delta(A_r))/Q(\Delta(A_r))| \rightarrow \infty$ as $r \rightarrow \infty$ (hence there are many intermediate lattices between P and Q), while $|P/Q|$ remains small for any Lie algebras of other types (see Bourbaki [2]).

Lastly in this section we quote here one more general result, which is a generalization and a refinement of (1.2).

Theorem 1. [17, Theorem 3.2] *For a compact connected semisimple Lie group G , let $\Delta = \Delta(G)$ be its root system, and $L = L(G)$ be its weight lattice. Let*

$\mathbf{k} = (k_\alpha)_{\alpha \in \Delta_+} \in \mathbb{N}^n$ ($n = |\Delta_+|$) satisfying $k_\alpha = k_\beta$ whenever $\|\alpha\| = \|\beta\|$. Let $\kappa = \sum_{\alpha \in \Delta_+} 2k_\alpha$. Then we have for $\nu \in P^\vee/Q^\vee$,

$$(3.16) \quad \begin{aligned} \zeta_r(2\mathbf{k}, \nu; G) &= \zeta_r(2\mathbf{k}, \nu; L; \Delta) \\ &= \frac{(-1)^n}{|W|} \left(\prod_{\alpha \in \Delta_+} \frac{(2\pi\sqrt{-1})^{2k_\alpha}}{(2k_\alpha)!} \right) \mathcal{P}(2\mathbf{k}, \nu; L; \Delta) \in \mathbb{Q} \cdot \pi^\kappa, \end{aligned}$$

where $\mathcal{P}(2\mathbf{k}, \nu; L; \Delta)$ is the Bernoulli function associated with L , defined in [17].

Note that when $L = P$, (3.16) coincides with our previous result in [13, Theorem 4.6].

As an example, here we apply this Theorem to the case of $PU(4)$.

Example 2. The generating function of $\mathcal{P}(\mathbf{k}, \mathbf{y}; A_3)$ has been given in [9, Example 2]. Therefore, by [17, (3.8)], we have

$$(3.17) \quad \begin{aligned} \mathcal{P}((2, 2, 2, 2, 2, 2), \mathbf{0}; Q; A_3) &= \frac{1}{4} (\mathcal{P}((2, 2, 2, 2, 2, 2), \mathbf{0}; A_3) - \mathcal{P}((2, 2, 2, 2, 2, 2), \lambda_1^\vee; A_3) \\ &\quad + \mathcal{P}((2, 2, 2, 2, 2, 2), \lambda_2^\vee; A_3) - \mathcal{P}((2, 2, 2, 2, 2, 2), \lambda_3^\vee; A_3)) \\ &= \frac{1103}{96888422400}, \end{aligned}$$

because $2\rho = 3\alpha_1 + 4\alpha_2 + 3\alpha_3$ and hence $\langle \mathbf{0}, 2\rho \rangle = 0$, $\langle \lambda_1^\vee, 2\rho \rangle = 3$, $\langle \lambda_2^\vee, 2\rho \rangle = 4$, $\langle \lambda_3^\vee, 2\rho \rangle = 3$. Therefore by Theorem 1, we obtain

$$(3.18) \quad \zeta_3((2, 2, 2, 2, 2, 2), \mathbf{0}; PU(4)) = \frac{1103\pi^{12}}{145332633600}.$$

4. SOME PREPARATORY LEMMAS

In this section, we give explicit functional relations for double polylogarithms (see Lemma 5 and Corollary 6). By use of these results, we give certain functional relations for triple zeta-functions of weight lattices in the next section.

First we quote the following two lemmas. Let $\phi(s) = \sum_{n \geq 1} (-1)^n n^{-s} = (2^{1-s} - 1) \zeta(s)$ and $\varepsilon_m = \{1 + (-1)^m\}/2$ for $m \in \mathbb{Z}$. Let $\{B_m(X)\}$ be the Bernoulli polynomials defined by

$$\frac{te^{Xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(X) \frac{t^m}{m!}.$$

Lemma 3. ([9, Lemma 9.1], [18, Lemma 2.1]) *Let $c \in [0, 2\pi) \subset \mathbb{R}$, and $h : \mathbb{N}_0 \rightarrow \mathbb{C}$ be a function (which may depend on c). Then, for $p \in \mathbb{N}$,*

$$(4.1) \quad \sum_{j=0}^p \phi(p-j) \varepsilon_{p-j} \sum_{\xi=0}^j h(j-\xi) \frac{(i(c-\pi))^\xi}{\xi!} = -\frac{1}{2} \sum_{\xi=0}^p h(p-\xi) \frac{(2\pi i)^\xi}{\xi!} B_\xi \left(\left\{ \frac{c}{2\pi} \right\} \right),$$

and

$$(4.2) \quad \sum_{j=0}^p \phi(p-j) \varepsilon_{p-j} \sum_{\xi=0}^j h(j-\xi) \frac{(i\pi)^\xi}{\xi!} = \sum_{\nu=0}^{[p/2]} \zeta(2\nu) h(p-2\nu) - \frac{i\pi}{2} h(p-1),$$

where $[x]$ is the integer part of $x \in \mathbb{R}$ and $\{x\} = x - [x]$.

The next lemma is the key to proving functional relations for zeta-functions. For $h \in \mathbb{N}$, let

$$\mathfrak{C} := \{C(l) \in \mathbb{C} \mid l \in \mathbb{Z}, l \neq 0\},$$

$$\mathfrak{D} := \{D(N; m; \eta) \in \mathbb{R} \mid N, m, \eta \in \mathbb{Z}, N \neq 0, m \geq 0, 1 \leq \eta \leq h\},$$

$$\mathfrak{A} := \{a_\eta \in \mathbb{N} \mid 1 \leq \eta \leq h\}$$

be sets of numbers indexed by integers, and let

$$\binom{x}{k} := \begin{cases} \frac{x(x-1)\cdots(x-k+1)}{k!} & (k \in \mathbb{N}), \\ 1 & (k = 0). \end{cases}$$

Lemma 4. [8, Lemma 6.2] *With the above notation, we assume that the infinite series appearing in*

$$(4.3) \quad \sum_{\substack{N \in \mathbb{Z} \\ N \neq 0}} (-1)^N C(N) e^{iN\theta} - 2 \sum_{\eta=1}^h \sum_{k=0}^{a_\eta} \phi(a_\eta - k) \varepsilon_{a_\eta - k} \\ \times \sum_{\xi=0}^k \left\{ \sum_{\substack{N \in \mathbb{Z} \\ N \neq 0}} (-1)^N D(N; k - \xi; \eta) e^{iN\theta} \right\} \frac{(i\theta)^\xi}{\xi!}$$

are absolutely convergent for $\theta \in [-\pi, \pi]$, and that (4.3) is a constant function for $\theta \in (-\pi, \pi)$. Then, for $d \in \mathbb{N}$,

(4.4)

$$\begin{aligned} \sum_{\substack{N \in \mathbb{Z} \\ N \neq 0}} \frac{(-1)^N C(N) e^{iN\theta}}{N^d} &= 2 \sum_{\eta=1}^h \sum_{k=0}^{a_\eta} \phi(a_\eta - k) \varepsilon_{a_\eta - k} \\ &\times \sum_{\xi=0}^k \left\{ \sum_{\omega=0}^{k-\xi} \binom{\omega + d - 1}{\omega} (-1)^\omega \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{(-1)^m D(m; k - \xi - \omega; \eta) e^{im\theta}}{m^{d+\omega}} \right\} \frac{(i\theta)^\xi}{\xi!} \\ &- 2 \sum_{k=0}^d \phi(d - k) \varepsilon_{d-k} \sum_{\xi=0}^k \left\{ \sum_{\eta=1}^h \sum_{\omega=0}^{a_\eta-1} \binom{\omega + k - \xi}{\omega} (-1)^\omega \right. \\ &\quad \times \left. \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{D(m; a_\eta - 1 - \omega; \eta)}{m^{k-\xi+\omega+1}} \right\} \frac{(i\theta)^\xi}{\xi!} \end{aligned}$$

holds for $\theta \in [-\pi, \pi]$, where the infinite series appearing on both sides of (4.4) are absolutely convergent for $\theta \in [-\pi, \pi]$.

For $p \in \mathbb{N}$, it is known that (see, for example, [8, (4.31), (4.32)])

$$(4.5) \quad \lim_{L \rightarrow \infty} \sum_{\substack{-L \leq l \leq L \\ l \neq 0}} \frac{(-1)^l e^{il\theta}}{l^p} = 2 \sum_{j=0}^p \phi(p-j) \varepsilon_{p-j} \frac{(i\theta)^j}{j!} \quad (\theta \in (-\pi, \pi)).$$

Note that the left-hand side is uniformly convergent for $\theta \in (-\pi, \pi)$ (see [24, § 3.35]), and is also absolutely convergent for $p \geq 2$. We prove the following lemma. Note that the case when p and q are even has been already proved in [8, (7.55)].

Lemma 5. For $p \in \mathbb{N}$, $s \in \mathbb{R}$ with $s > 1$ and $x \in \mathbb{C}$ with $|x| = 1$,

$$(4.6) \quad \begin{aligned} & \sum_{\substack{l \neq 0, m \geq 1 \\ l+m \neq 0}} \frac{(-1)^{l+m} x^m e^{i(l+m)\theta}}{l^p m^s (l+m)^q} \\ & - 2 \sum_{j=0}^p \phi(p-j) \varepsilon_{p-j} \sum_{\xi=0}^j \binom{q-1+j-\xi}{q-1} (-1)^{j-\xi} \sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{im\theta}}{m^{s+q+j-\xi}} \frac{(i\theta)^\xi}{\xi!} \\ & + 2 \sum_{j=0}^q \phi(q-j) \varepsilon_{q-j} \sum_{\xi=0}^j \binom{p-1+j-\xi}{p-1} (-1)^{p-1} \sum_{m=1}^{\infty} \frac{x^m}{m^{s+p+j-\xi}} \frac{(i\theta)^\xi}{\xi!} = 0 \end{aligned}$$

for $\theta \in [-\pi, \pi]$.

Proof. First we assume $p \geq 2$. Then, for $\theta \in (-\pi, \pi)$, it follows from (4.5) that

$$(4.7) \quad \left(\sum_{\substack{l \in \mathbb{Z} \\ l \neq 0}} \frac{(-1)^l e^{il\theta}}{l^p} - 2 \sum_{j=0}^p \phi(p-j) \varepsilon_{p-j} \frac{(i\theta)^j}{j!} \right) \sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{im\theta}}{m^s} = 0,$$

where the left-hand side is absolutely and uniformly convergent for $\theta \in (-\pi, \pi)$.

Therefore we have

$$(4.8) \quad \begin{aligned} & \sum_{\substack{l \in \mathbb{Z}, l \neq 0 \\ m \geq 1 \\ l+m \neq 0}} \frac{(-1)^{l+m} x^m e^{i(l+m)\theta}}{l^p m^s} - 2 \sum_{j=0}^p \phi(p-j) \varepsilon_{p-j} \left\{ \sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{im\theta}}{m^s} \right\} \frac{(i\theta)^j}{j!} \\ & = (-1)^{p+1} \sum_{m=1}^{\infty} \frac{x^m}{m^{s+p}} \end{aligned}$$

for $\theta \in (-\pi, \pi)$. Now we use Lemma 4 with $h = 1$, $a_1 = p$,

$$C(N) = \sum_{\substack{l \neq 0, m \geq 1 \\ l+m=N}} \frac{x^m}{l^p m^s} \quad (N \in \mathbb{Z}, N \neq 0)$$

and $D(N; \mu; 1) = x^N N^{-s}$ (if $\mu = 0$ and $N \geq 1$), or $= 0$ (otherwise). Under these settings, we see that the left-hand side of (4.8) is of the form (4.3). Furthermore

the right-hand side of (4.8) is a constant as a function in θ . Therefore we can apply Lemma 4 with $d = q \in \mathbb{N}$. Then (4.4) gives (4.6) for $p \geq 2$.

Next we prove the case $p = 1$. As we proved above, (4.6) in the case $p = 2$ holds. Replacing x by $-xe^{i\theta}$ in this case, we have

$$(4.9) \quad \sum_{\substack{l \neq 0, m \geq 1 \\ l+m \neq 0}} \frac{(-1)^l x^m e^{il\theta}}{l^2 m^s (l+m)^q} \\ - 2 \sum_{j=0}^2 \phi(2-j) \varepsilon_{2-j} \sum_{\xi=0}^j \binom{q-1+j-\xi}{q-1} (-1)^{j-\xi} \sum_{m=1}^{\infty} \frac{x^m}{m^{s+q+j-\xi}} \frac{(i\theta)^\xi}{\xi!} \\ + 2 \sum_{j=0}^q \phi(q-j) \varepsilon_{q-j} \sum_{\xi=0}^j \binom{1+j-\xi}{1} (-1)^1 \sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{-im\theta}}{m^{s+2+j-\xi}} \frac{(i\theta)^\xi}{\xi!} = 0$$

for $\theta \in [-\pi, \pi]$. We denote the first, the second and the third term on the left hand side of (4.9) by $I_1(\theta)$, $I_2(\theta)$ and $I_3(\theta)$, respectively. We differentiate these terms in θ . We can easily compute $I_1'(\theta)$ and $I_2'(\theta)$. As for $I_3'(\theta)$, we have

$$I_3'(\theta) = 2 \sum_{j=0}^q \phi(q-j) \varepsilon_{q-j} \left\{ -i \sum_{\xi=0}^j (1+j-\xi) (-1) \sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{-im\theta}}{m^{s+1+j-\xi}} \frac{(i\theta)^\xi}{\xi!} \right. \\ \left. + i \sum_{\xi=1}^j (1+j-\xi) (-1) \sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{-im\theta}}{m^{s+2+j-\xi}} \frac{(i\theta)^{\xi-1}}{(\xi-1)!} \right\}.$$

Note that as for the second member in the curly brackets on the right-hand side, ξ may also run from 1 to $j+1$ because $1+j-(j+1) = 0$ in the summand. Hence, by replacing $\xi-1$ by ξ , we have

$$I_3'(\theta) = 2i \sum_{j=0}^q \phi(q-j) \varepsilon_{q-j} \sum_{\xi=0}^j \sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{-im\theta}}{m^{s+1+j-\xi}} \frac{(i\theta)^\xi}{\xi!}.$$

Thus, we see that $(I_1'(\theta) + I_2'(\theta) + I_3'(\theta))/i$, replacing x by $-xe^{i\theta}$, gives (4.6) in the case $p = 1$. This completes the proof. \square

The following special cases of (4.6) will be necessary in the next section.

Corollary 6. For $p, q \in \mathbb{N}$, $s > 1$ and $x, y \in \mathbb{C}$ with $|x| \leq 1$ and $|y| \leq 1$, let

$$(4.10) \quad \mathfrak{T}(p, s, q; x, y) = \sum_{\substack{l \neq 0, m \geq 1 \\ l+m \neq 0}} \frac{x^l y^m}{l^p m^s (l+m)^q}.$$

Then

$$(4.11) \quad \mathfrak{T}(p, s, q; 1, 1) = 2(-1)^p \sum_{k=0}^{[p/2]} \zeta(2k) \binom{p+q-1-2k}{q-1} \zeta(s+p+q-2k) \\ + 2(-1)^p \sum_{k=0}^{[q/2]} \zeta(2k) \binom{p+q-1-2k}{p-1} \zeta(s+p+q-2k),$$

$$(4.12) \quad \mathfrak{T}(p, s, q; -1, 1) = 2(-1)^p \sum_{k=0}^{[p/2]} \phi(2k) \binom{p+q-1-2k}{q-1} \zeta(s+p+q-2k) \\ + 2(-1)^p \sum_{k=0}^{[q/2]} \phi(2k) \binom{p+q-1-2k}{p-1} \phi(s+p+q-2k),$$

$$(4.13) \quad \mathfrak{T}(p, s, q; 1, -1) = 2(-1)^p \sum_{k=0}^{[p/2]} \zeta(2k) \binom{p+q-1-2k}{q-1} \phi(s+p+q-2k) \\ + 2(-1)^p \sum_{k=0}^{[q/2]} \zeta(2k) \binom{p+q-1-2k}{p-1} \phi(s+p+q-2k),$$

$$(4.14) \quad \mathfrak{T}(p, s, q; -1, -1) = 2(-1)^p \sum_{k=0}^{[p/2]} \phi(2k) \binom{p+q-1-2k}{q-1} \phi(s+p+q-2k) \\ + 2(-1)^p \sum_{k=0}^{[q/2]} \phi(2k) \binom{p+q-1-2k}{p-1} \zeta(s+p+q-2k).$$

Proof. We can directly obtain (4.14) and (4.12) by letting $(x, \theta) = (1, 0)$, $(-1, 0)$, respectively in (4.6). (Since $\theta = 0$, all the terms corresponding to $\xi \geq 1$ vanish on the right-hand side of (4.6).) As for (4.13) and (4.11), we let $(x, \theta) = (-1, \pi)$, $(1, \pi)$, respectively in (4.6) and use Lemma 3. \square

Remark 7. The results of the above corollary are not new. The formula (4.11) was first proved in [23], and then, by a different method, Nakamura [20, Theorem 3.1] has shown all of the above (see also a survey given in [8, Section 3]).

5. THE ZETA-FUNCTION OF $SU(4)$

Now we start to prove functional relations for zeta-functions for lattices of type A_3 . We use the same technique as in our previous paper [8, Section 7]. Hence the details of their proofs will be omitted.

In this section we study the zeta-function associated with the group $SU(4)$. Our starting point is similar to (4.7), or [8, Equation (7.58)]. We begin by considering

$$(5.1) \quad \left(\sum_{\substack{l \in \mathbb{Z} \\ l \neq 0}} \frac{(-1)^l e^{il\theta}}{l^p} - 2 \sum_{j=0}^p \phi(p-j) \varepsilon_{p-j} \frac{(i\theta)^j}{j} \right) \sum_{\substack{m \in \mathbb{Z}, m \neq 0 \\ n \geq 1 \\ m+n \neq 0}} \frac{(-1)^{m+n} x^m y^n e^{i(m+n)\theta}}{m^q n^s (m+n)^b} = 0$$

for $\theta \in [-\pi, \pi]$, where $p, q, b \in \mathbb{N}$ with $p \geq 2$, $s \in \mathbb{R}$ with $s > 1$ and $x, y \in \mathbb{C}$ with $|x| = |y| = 1$. Then, by the (almost) same argument as in [8, pp.158-160], we

obtain

$$\begin{aligned}
(5.2) \quad & \sum_{\substack{l, m \neq 0, n \geq 1 \\ l+m \neq 0, m+n \neq 0 \\ l+m+n \neq 0}} \frac{(-1)^{l+m} x^m y^n e^{i(l+m)\theta}}{l^p m^q n^s (l+m)^a (m+n)^b (l+m+n)^c} \\
&= 2 \sum_{k=0}^p \phi(p-k) \varepsilon_{p-k} \sum_{\xi=0}^k \sum_{\omega=0}^{k-\xi} \binom{\omega+a-1}{\omega} (-1)^\omega \binom{k-\xi-\omega+c-1}{k-\xi-\omega} \\
&\quad \times (-1)^{k-\xi-\omega} \sum_{\substack{m \neq 0 \\ n \geq 1 \\ m+n \neq 0}} \frac{(-1)^m x^m y^n e^{im\theta}}{m^{q+a+\omega} n^s (m+n)^{b+c+k-\xi-\omega}} \frac{(i\theta)^\xi}{\xi!} \\
&\quad - 2 \sum_{j=0}^c \phi(c-k) \varepsilon_{c-k} \sum_{\xi=0}^k \sum_{\omega=0}^{k-\xi} \binom{\omega+a-1}{\omega} (-1)^\omega \binom{k-\xi-\omega+p-1}{p-1} \\
&\quad \times (-1)^{p-1+a+\omega} \sum_{\substack{m \neq 0 \\ n \geq 1 \\ m+n \neq 0}} \frac{(-1)^n x^m y^n e^{-in\theta}}{m^q n^{s+a+\omega} (m+n)^{p+b+k-\xi-\omega}} \frac{(i\theta)^\xi}{\xi!} \\
&\quad - 2 \sum_{k=0}^a \phi(a-k) \varepsilon_{a-k} \sum_{\xi=0}^k \sum_{\omega=0}^{p-1} \binom{\omega+k-\xi}{\omega} (-1)^\omega \binom{p+c-2-\omega}{p-1-\omega} \\
&\quad \times (-1)^{p-1-\omega} \sum_{\substack{m \neq 0 \\ n \geq 1 \\ m+n \neq 0}} \frac{x^m y^n}{m^{q+k-\xi+\omega+1} n^s (m+n)^{p+b+c-1-\omega}} \frac{(i\theta)^\xi}{\xi!} \\
&\quad + 2 \sum_{k=0}^a \phi(a-k) \varepsilon_{a-k} \sum_{\xi=0}^k \sum_{\omega=0}^{c-1} \binom{\omega+k-\xi}{\omega} (-1)^\omega \binom{p+c-2-\omega}{p-1} \\
&\quad \times (-1)^{p+k-\xi+\omega} \sum_{\substack{m \neq 0 \\ n \geq 1 \\ m+n \neq 0}} \frac{x^m y^n}{m^q n^{s+k-\xi+\omega+1} (m+n)^{p+b+c-1-\omega}} \frac{(i\theta)^\xi}{\xi!}
\end{aligned}$$

for $\theta \in [-\pi, \pi]$ and $p, q, a, b, c \in \mathbb{N}$. (A small difference is that, in the course of the argument, we replaced x by $-e^{-i\theta}$ in [8], while this time we replace y by $-ye^{-i\theta}$.) Note that (5.2) in the case $p = 1$ can be proved similarly to Lemma 5.

Now we put $(x, y, \theta) = (-1, -1, 0)$ in (5.2), namely we take notice of the constant term of (5.2). We proceed similarly to the argument in [8]; that is, we decompose the left-hand side of (5.2) by the method written in [8, p. 160], while apply (4.2) to the right-hand side. Then we obtain the following theorem.

Theorem 8. For $p, q, a, b, c \in \mathbb{N}$,

(5.3)

$$\begin{aligned}
& \zeta_3((p, q, s, a, b, c), \lambda_2^\vee; SU(4)) + (-1)^p \zeta_3((p, a, s, q, c, b), \lambda_2^\vee; SU(4)) \\
& + (-1)^{p+a} \zeta_3((q, a, c, p, s, b), \lambda_2^\vee; SU(4)) + (-1)^{p+a+c} \zeta_3((q, s, c, b, a, p), \lambda_2^\vee; SU(4)) \\
& + (-1)^q \zeta_3((a, q, b, p, s, c), \lambda_2^\vee; SU(4)) + (-1)^{q+b} \zeta_3((a, s, b, c, q, p), \lambda_2^\vee; SU(4))
\end{aligned}$$

$$\begin{aligned}
& + (-1)^{q+a} \zeta_3((a, p, b, q, c, s), \lambda_2^\vee; SU(4)) + (-1)^{q+a+b} \zeta_3((a, c, b, s, p, q), \lambda_2^\vee; SU(4)) \\
& + (-1)^{q+a+b+c} \zeta_3((s, c, p, a, b, q), \lambda_2^\vee; SU(4)) + (-1)^{p+q+a} \zeta_3((q, p, c, a, b, s), \lambda_2^\vee; SU(4)) \\
& + (-1)^{p+q+a+c} \zeta_3((q, b, c, s, p, a), \lambda_2^\vee; SU(4)) \\
& + (-1)^{p+q+a+b+c} \zeta_3((s, b, p, q, c, a), \lambda_2^\vee; SU(4)) \\
& = 2(-1)^p \sum_{j=0}^{[p/2]} \phi(2j) \sum_{\omega=0}^{p-2j} \binom{\omega+a-1}{\omega} \binom{p+c-2j-\omega-1}{c-1} \\
& \quad \times \mathfrak{T}(q+a+\omega, s, p+b+c-2j-\omega; 1, -1) \\
& + 2(-1)^{p+a} \sum_{j=0}^{[c/2]} \phi(2j) \sum_{\omega=0}^{c-2j} \binom{\omega+a-1}{\omega} \binom{p+c-2j-\omega-1}{p-1} \\
& \quad \times \mathfrak{T}(q, s+a+\omega, p+b+c-2j-\omega; -1, 1) \\
& + 2(-1)^p \sum_{j=0}^{[a/2]} \phi(2j) \sum_{\omega=0}^{p-1} \binom{\omega+a-2j}{\omega} \binom{p+c-2-\omega}{c-1} \\
& \quad \times \mathfrak{T}(q+a-2j+\omega+1, s, p+b+c-1-\omega; -1, -1) \\
& + 2(-1)^{p+a} \sum_{j=0}^{[a/2]} \phi(2j) \sum_{\omega=0}^{c-1} \binom{\omega+a-2j}{\omega} \binom{p+c-2-\omega}{p-1} \\
& \quad \times \mathfrak{T}(q, s+a-2j+\omega+1, p+b+c-1-\omega; -1, -1)
\end{aligned}$$

holds for $s \in \mathbb{C}$ except for singularities of functions on both sides, where $\mathfrak{T}(p, s, q; x, y)$ is defined by (4.10). Moreover, from (4.11)-(4.14) we see that the right-hand side of the above can be written in terms of the Riemann zeta-function.

Setting $(a, b, c, p, q, s) = (2k, 2k, 2k, 2k, 2k, 2k)$ for $k \in \mathbb{N}$, we obtain

$$\zeta_3((2k, 2k, 2k, 2k, 2k, 2k), \lambda_2^\vee; SU(4)) \in \mathbb{Q} \cdot \pi^{12k},$$

and the rational coefficients can be determined explicitly. This gives an example of [17, Theorem 3.2].

Similarly, by putting $(x, y, \theta) = (1, 1, \pi)$ in (5.2) and using (4.2) in Lemma 3, we obtain the following theorem for $\zeta_3(\mathbf{s}; A_3) = \zeta_3(\mathbf{s}, \mathbf{0}; SU(4))$ (see (3.7)).

Theorem 9. For $p, q, a, b, c \in \mathbb{N}$,

$$\begin{aligned}
(5.4) \quad & \zeta_3((p, q, s, a, b, c); A_3) + (-1)^p \zeta_3((p, a, s, q, c, b); A_3) \\
& + (-1)^{p+a} \zeta_3((q, a, c, p, s, b); A_3) + (-1)^{p+a+c} \zeta_3((q, s, c, b, a, p); A_3) \\
& + (-1)^q \zeta_3((a, q, b, p, s, c); A_3) + (-1)^{q+b} \zeta_3((a, s, b, c, q, p); A_3) \\
& + (-1)^{q+a} \zeta_3((a, p, b, q, c, s); A_3) + (-1)^{q+a+b} \zeta_3((a, c, b, s, p, q); A_3) \\
& + (-1)^{q+a+b+c} \zeta_3((s, c, p, a, b, q); A_3) + (-1)^{p+q+a} \zeta_3((q, p, c, a, b, s); A_3) \\
& + (-1)^{p+q+a+c} \zeta_3((q, b, c, s, p, a); A_3) + (-1)^{p+q+a+b+c} \zeta_3((s, b, p, q, c, a); A_3)
\end{aligned}$$

$$\begin{aligned}
&= 2(-1)^p \sum_{j=0}^{[p/2]} \zeta(2j) \sum_{\omega=0}^{p-2j} \binom{\omega+a-1}{\omega} \binom{p+c-2j-\omega-1}{c-1} \\
&\quad \times \mathfrak{T}(q+a+\omega, s, p+b+c-2j-\omega; 1, 1) \\
&+ 2(-1)^{p+a} \sum_{j=0}^{[c/2]} \zeta(2j) \sum_{\omega=0}^{c-2j} \binom{\omega+a-1}{\omega} \binom{p+c-2j-\omega-1}{p-1} \\
&\quad \times \mathfrak{T}(q, s+a+\omega, p+b+c-2j-\omega; 1, 1) \\
&+ 2(-1)^p \sum_{j=0}^{[a/2]} \zeta(2j) \sum_{\omega=0}^{p-1} \binom{\omega+a-2j}{\omega} \binom{p+c-2-\omega}{c-1} \\
&\quad \times \mathfrak{T}(q+a-2j+\omega+1, s, p+b+c-1-\omega; 1, 1) \\
&+ 2(-1)^{p+a} \sum_{j=0}^{[a/2]} \zeta(2j) \sum_{\omega=0}^{c-1} \binom{\omega+a-2j}{\omega} \binom{p+2-2-\omega}{p-1} \\
&\quad \times \mathfrak{T}(q, s+a-2j+\omega+1, p+b+c-\omega-1; 1, 1)
\end{aligned}$$

holds for $s \in \mathbb{C}$ except for singularities of functions on both sides.

When p, q, a, b, c are all even, this theorem has already been proved in [8, Theorem 7.1]. The expression of the left-hand side in [8] is a little different from the above, but we can easily check that those two expressions are equal, using (3.15).

Setting $(a, b, c, p, q, s) = (2k, 2k, 2k, 2k, 2k, 2k)$ for $k \in \mathbb{N}$, we obtain (1.2) for A_3 with the explicit value of the coefficient. On the other hand, when $p = q = a = b = c$ which is an odd integer, the left-hand sides of the above two theorems are equal to 0. This is because $\zeta_3((p, p, p, s, p, p); A_3) = \zeta_3((p, p, p, p, s, p); A_3)$ and $\zeta_3((p, p, p, s, p, p), \lambda_2^\vee; SU(4)) = \zeta_3((p, p, p, p, s, p), \lambda_2^\vee; SU(4))$, by (3.14), (3.15). Hence, unfortunately we can obtain no information about, for example, $\zeta_3((2k+1, 2k+1, 2k+1, 2k+1, 2k+1, 2k+1); A_3)$ and $\zeta_3(((2k+1, 2k+1, 2k+1, 2k+1, 2k+1, 2k+1)), \lambda_2^\vee; SU(4))$ ($k \in \mathbb{N}_0$) from the above theorems.

However, choosing (a, b, c, p, q, s) suitably, we can obtain some classes of evaluation formulas for them. For example, set

$$(a, b, c, p, q, s) = (2k+1, 2k+1, 2k+1, 2k+1, 2k, 2k+1) \quad (k \in \mathbb{N})$$

in (5.3) and (5.4). Then the left-hand sides of them are

$$\begin{aligned}
&2\zeta_3((2k+1, 2k, 2k+1, 2k+1, 2k+1, 2k+1), \lambda_2^\vee; SU(4)) \\
&\quad - 2\zeta_3((2k+1, 2k+1, 2k+1, 2k, 2k+1, 2k+1), \lambda_2^\vee; SU(4)) \\
&\quad = -2\zeta_3((2k, 2k+1, 2k+1, 2k+1, 2k+1, 2k+1), \lambda_2^\vee; SU(4)), \\
&2\zeta_3((2k+1, 2k, 2k+1, 2k+1, 2k+1, 2k+1); A_3) \\
&\quad - 2\zeta_3((2k+1, 2k+1, 2k+1, 2k, 2k+1, 2k+1); A_3)
\end{aligned}$$

$$= -2\zeta_3((2k, 2k+1, 2k+1, 2k+1, 2k+1, 2k+1); A_3),$$

respectively, by using the relation

$$\frac{1}{l^{2k+1}m^{2k}(l+m)^{2k+1}} - \frac{1}{l^{2k+1}m^{2k+1}(l+m)^{2k}} = -\frac{1}{l^{2k}m^{2k+1}(l+m)^{2k+1}}.$$

Therefore we obtain the following.

Proposition 10. *For $k \in \mathbb{N}$,*

$$\zeta_3((2k, 2k+1, 2k+1, 2k+1, 2k+1, 2k+1), \lambda_2^\vee; SU(4)) \in \mathbb{Q}[\{\zeta(j) \mid j \in \mathbb{N}_{>1}\}],$$

$$\zeta_3((2k, 2k+1, 2k+1, 2k+1, 2k+1, 2k+1); A_3) \in \mathbb{Q}[\{\zeta(j) \mid j \in \mathbb{N}_{>1}\}],$$

and the rational coefficients can be determined explicitly.

Example 11. Setting $(a, b, c, p, q, s) = (2k+1, 2k+1, 2k+1, 2k+1, 2k, 2k+1)$ in (5.4), we can obtain

$$\zeta_3(2, 3, 3, 3, 3, 3; A_3) = \frac{\pi^6}{63} \zeta(11) + \frac{199\pi^4}{30} \zeta(13) - 365\pi^2 \zeta(15) + 2941 \zeta(17),$$

$$\zeta_3(4, 5, 5, 5, 5, 5; A_3)$$

$$= \frac{152\pi^{12}}{18243225} \zeta(17) + \frac{17\pi^{10}}{6237} \zeta(19) + \frac{29\pi^8}{54} \zeta(21) + \frac{979\pi^6}{9} \zeta(23) \\ + \frac{15585\pi^4}{2} \zeta(25) - 660975\pi^2 \zeta(27) + 5654565 \zeta(29),$$

$$\zeta_3(6, 7, 7, 7, 7, 7; A_3)$$

$$= \frac{2062\pi^{18}}{506224616625} \zeta(23) + \frac{11776\pi^{16}}{5367718125} \zeta(25) + \frac{8\pi^{14}}{13365} \zeta(27) \\ + \frac{10223594\pi^{12}}{91216125} \zeta(29) + \frac{103486\pi^{10}}{6075} \zeta(31) + \frac{5459978\pi^8}{2025} \zeta(33) \\ + \frac{3464974\pi^6}{15} \zeta(35) + \frac{41963621\pi^4}{3} \zeta(37) - 1456076440\pi^2 \zeta(39) \\ + 12758984832 \zeta(41),$$

$$\zeta_3(8, 9, 9, 9, 9, 9; A_3)$$

$$= \frac{64586\pi^{24}}{37355158168453125} \zeta(29) + \frac{422704\pi^{22}}{298841265347625} \zeta(31) \\ + \frac{10664\pi^{20}}{19088409375} \zeta(33) + \frac{663259\pi^{18}}{4632120675} \zeta(35) + \frac{2307883\pi^{16}}{84341250} \zeta(37) \\ + \frac{6327646\pi^{14}}{1488375} \zeta(39) + \frac{860790601\pi^{12}}{1488375} \zeta(41) + \frac{380997529\pi^{10}}{4725} \zeta(43) \\ + \frac{7867619353\pi^8}{1050} \zeta(45) + \frac{164035120733\pi^6}{315} \zeta(47) + \frac{59740238129\pi^4}{2} \zeta(49) \\ - 3514635376395\pi^2 \zeta(51) + 31198575194215 \zeta(53),$$

$$\zeta_3(10, 11, 11, 11, 11, 11; A_3)$$

$$= \frac{221912776\pi^{30}}{332660210652234981140625} \zeta(35) + \frac{10705232\pi^{28}}{13854831558583640625} \zeta(37)$$

$$\begin{aligned}
& + \frac{5135896\pi^{26}}{12250072111921875} \zeta(39) + \frac{4767865562\pi^{24}}{33250195732359375} \zeta(41) + \frac{222974564\pi^{22}}{6269397175125} \zeta(43) \\
& + \frac{24806393774\pi^{20}}{3569532553125} \zeta(45) + \frac{2589565814\pi^{18}}{2290609125} \zeta(47) + \frac{1188339011\pi^{16}}{7441875} \zeta(49) \\
& + \frac{3650193872\pi^{14}}{178605} \zeta(51) + \frac{11782765221344\pi^{12}}{4465125} \zeta(53) + \frac{35232949154\pi^{10}}{135} \zeta(55) \\
& + \frac{18601660627979\pi^8}{945} \zeta(57) + \frac{393366314952754\pi^6}{315} \zeta(59) \\
& + \frac{1050680447134747\pi^4}{15} \zeta(61) - 8947964548486678\pi^2 \zeta(63) \\
& + 80075393000830422 \zeta(65).
\end{aligned}$$

Also, setting $(a, b, c, p, q, s) = (2k+1, 2k+1, 2k+1, 2k+1, 2k, 2k+1)$ in (5.3), we can obtain

$$\begin{aligned}
& \zeta_3((2, 3, 3, 3, 3, 3), \lambda_2^\vee; SU(4)) \\
& = \frac{17\pi^8}{344064} \zeta(9) + \frac{22847\pi^6}{1720320} \zeta(11) + \frac{49005\pi^4}{16384} \zeta(13) + \frac{3768307\pi^2}{98304} \zeta(15) - \frac{11189819}{16384} \zeta(17), \\
& \zeta_3((4, 5, 5, 5, 5, 5), \lambda_2^\vee; SU(4)) \\
& = \frac{693547\pi^{14}}{51011754393600} \zeta(15) + \frac{714624223\pi^{12}}{81618807029760} \zeta(17) + \frac{28726157\pi^{10}}{11072962560} \zeta(19) \\
& + \frac{25906094783\pi^8}{54358179840} \zeta(21) + \frac{9177921545\pi^6}{113246208} \zeta(23) + \frac{2422120970909\pi^4}{671088640} \zeta(25) \\
& + \frac{7798050014825\pi^2}{134217728} \zeta(27) - \frac{270498379148235}{268435456} \zeta(29), \\
& \zeta_3((6, 7, 7, 7, 7, 7), \lambda_2^\vee; SU(4)) \\
& = \frac{2752145869\pi^{20}}{773055350341160140800} \zeta(21) + \frac{1098434242057681\pi^{18}}{255108265612582846464000} \zeta(23) \\
& + \frac{150866953637\pi^{16}}{68882685493248000} \zeta(25) + \frac{20612241204619\pi^{14}}{34824024332697600} \zeta(27) \\
& + \frac{19614225808011463\pi^{12}}{179094982282444800} \zeta(29) + \frac{6776217678200971\pi^{10}}{417470821171200} \zeta(31) \\
& + \frac{337234670875566533\pi^8}{139156940390400} \zeta(33) + \frac{7289362333395816433\pi^6}{43293270343680} \zeta(35) \\
& + \frac{3412540143011100899\pi^4}{515396075520} \zeta(37) + \frac{32450037853433343325\pi^2}{274877906944} \zeta(39) \\
& - \frac{274409134558621990125}{137438953472} \zeta(41).
\end{aligned}$$

The authors also checked, by using Mathematica 8, that the above formulas agree with numerical computation, based on the definitions of zeta-functions.

6. THE ZETA-FUNCTION OF $SO(6)$

Next we consider $\zeta_3(\mathbf{s}; SO(6))$. It follows from (3.7) and (3.8) that

$$(6.1) \quad \zeta_3(\mathbf{s}; SO(6)) = \frac{1}{2} \{ \zeta_3(\mathbf{s}; A_3) + \zeta_3(\mathbf{s}, \lambda_2^\vee; SU(4)) \}.$$

Combining Theorems 8 and 9 and using (6.1), we can obtain functional relations among $\zeta_3(\mathbf{s}; SO(6))$ and $\zeta(s)$.

Theorem 12. For $p, q, a, b, c \in \mathbb{N}$,

$$(6.2) \quad \begin{aligned} & \zeta_3((p, q, s, a, b, c); SO(6)) + (-1)^p \zeta_3((p, a, s, q, c, b); SO(6)) \\ & + (-1)^{p+a} \zeta_3((q, a, c, p, s, b); SO(6)) + (-1)^{p+a+c} \zeta_3((q, s, c, b, a, p); SO(6)) \\ & + (-1)^q \zeta_3((a, q, b, p, s, c); SO(6)) + (-1)^{q+b} \zeta_3((a, s, b, c, q, p); SO(6)) \\ & + (-1)^{q+a} \zeta_3((a, p, b, q, c, s); SO(6)) + (-1)^{q+a+b} \zeta_3((a, c, b, s, p, q); SO(6)) \\ & + (-1)^{q+a+b+c} \zeta_3((s, c, p, a, b, q); SO(6)) + (-1)^{p+q+a} \zeta_3((q, p, c, a, b, s); SO(6)) \\ & + (-1)^{p+q+a+c} \zeta_3((q, b, c, s, p, a); SO(6)) + (-1)^{p+q+a+b+c} \zeta_3((s, b, p, q, c, a); SO(6)) \\ & = \frac{1}{2} (J_0 + J_2) \end{aligned}$$

holds for $s \in \mathbb{C}$ except for singularities of functions on both sides, where J_0 and J_2 are the right-hand sides of (5.4) and (5.3), respectively.

Similarly to Proposition 10 and Example 11, we can obtain the following.

Proposition 13. For $k \in \mathbb{N}$,

$$\begin{aligned} & \zeta_3((2k, 2k, 2k, 2k, 2k, 2k); SO(6)) \in \mathbb{Q} \cdot \pi^{12k}, \\ & \zeta_3((2k, 2k+1, 2k+1, 2k+1, 2k+1, 2k+1); SO(6)) \in \mathbb{Q}[\{\zeta(j) \mid j \in \mathbb{N}_{>1}\}], \end{aligned}$$

and the rational coefficients can be determined explicitly.

Example 14. Setting $(a, b, c, p, q, s) = (2, 2, 2, 2, 2, s)$ in (6.2), we obtain

$$(6.3) \quad \begin{aligned} & 2\zeta_3((2, s, 2, 2, 2, 2); SO(6)) + 4\zeta_3((2, 2, s, 2, 2, 2); SO(6)) \\ & + 4\zeta_3((2, 2, 2, s, 2, 2); SO(6)) + 2\zeta_3((2, 2, 2, 2, s, 2); SO(6)) \\ & = (93 \cdot 2^{-s-8} + 306) \zeta(s+10) + (3 \cdot 2^{-s-4} - 260) \zeta(2) \zeta(s+8) \\ & - (67 \cdot 2^{-s-6} - 110) \zeta(4) \zeta(s+6) - \frac{1}{8} (5 \cdot 2^{-s-3} - 21) \zeta(6) \zeta(s+4). \end{aligned}$$

In particular, when $s = 2$ in (6.3), we have

$$(6.4) \quad \zeta_3((2, 2, 2, 2, 2, 2); SO(6)) = \frac{10411}{1307674368000} \pi^{12}.$$

Also, combining (6.1) and the results in Example 11, we obtain, for example,

(6.5)

$$\begin{aligned} \zeta_3((2, 3, 3, 3, 3, 3); SO(6)) = & \frac{17\pi^8}{688128}\zeta(9) + \frac{150461\pi^6}{10321920}\zeta(11) + \frac{2365283\pi^4}{491520}\zeta(13) \\ & - \frac{32112653\pi^2}{196608}\zeta(15) + \frac{36995525}{32768}\zeta(17). \end{aligned}$$

7. THE ZETA-FUNCTION OF $PU(4)$

Finally we consider the case of the group $PU(4)$. An interesting feature in this case is the appearance of a Dirichlet L -function, so we will describe some details of the argument.

First we slightly generalize the results used in the previous sections. Let

$$\phi(s; \alpha) = \sum_{m=1}^{\infty} e^{2m\pi i \alpha} m^{-s}$$

be the Lerch zeta-function for $\alpha \in \mathbb{R}$. We can easily see that $\phi(s; 1/2)$ is equal to $\phi(s) = (2^{1-s} - 1) \zeta(s)$ used in Section 4, and

$$(7.1) \quad \phi(s; 1/4) = 2^{-s} (2^{1-s} - 1) \zeta(s) + iL(s, \chi_4),$$

$$(7.2) \quad \phi(s; -1/4) = 2^{-s} (2^{1-s} - 1) \zeta(s) - iL(s, \chi_4),$$

where $L(s, \chi_4) = \sum_{m \geq 0} (-1)^m (2m+1)^{-s}$ be the Dirichlet L -function associated with the primitive Dirichlet character χ_4 of conductor 4. Moreover we let

$$\Lambda(s; i) = \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{i^m}{m^s} = 2^{-s} (1 + e^{-\pi i s}) (2^{1-s} - 1) \zeta(s) + i (1 - e^{-\pi i s}) L(s, \chi_4),$$

$$\Lambda(s; -i) = \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{(-i)^m}{m^s} = 2^{-s} (1 + e^{-\pi i s}) (2^{1-s} - 1) \zeta(s) - i (1 - e^{-\pi i s}) L(s, \chi_4),$$

where $m^{-s} = \exp(-s(\log |m| + \pi i))$ for $m < 0$. In particular, for $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$, we have

$$(7.3) \quad \Lambda(2k; i) = 2^{1-2k} (2^{1-2k} - 1) \zeta(2k); \quad \Lambda(2l+1; i) = 2iL(2l+1, \chi_4),$$

$$(7.4) \quad \Lambda(2k; -i) = 2^{1-2k} (2^{1-2k} - 1) \zeta(2k); \quad \Lambda(2l+1; -i) = -2iL(2l+1, \chi_4).$$

Also, it is well-known that

$$(7.5) \quad \lim_{K \rightarrow \infty} \sum_{\substack{k=-K \\ k \neq 0}}^K \frac{e^{2\pi i k \alpha}}{k^j} = -B_j(\alpha) \frac{(2\pi i)^j}{j!} \quad (j \in \mathbb{N}; \alpha \in [0, 1))$$

(see, for example, [1, Theorem 12.19]). Here, setting $c = \pi/2$ and $3\pi/2$ in (4.1) and using (7.5) with $\alpha = \pm 1/4$, we obtain the following.

Lemma 15. For any $p \in \mathbb{N}$ and any function $h : \mathbb{N}_0 \rightarrow \mathbb{C}$,

$$(7.6) \quad \sum_{j=0}^p \phi(p-j) \varepsilon_{p-j} \sum_{\xi=0}^j h(j-\xi) \frac{(-i\pi/2)^\xi}{\xi!} = \frac{1}{2} \sum_{\xi=0}^p \Lambda(\xi; i) h(p-\xi),$$

$$(7.7) \quad \sum_{j=0}^p \phi(p-j) \varepsilon_{p-j} \sum_{\xi=0}^j h(j-\xi) \frac{(i\pi/2)^\xi}{\xi!} = \frac{1}{2} \sum_{\xi=0}^p \Lambda(\xi; -i) h(p-\xi).$$

Set $(x, \theta) = (-i, 0), (-i, \pi/2), (-1, \pi/2), (i, 0), (i, -\pi/2), (-1, -\pi/2)$ in (4.6) and use Lemma 15. Then, by the same method as in the proof of Corollary 6, we obtain the following.

Lemma 16. For $p, q \in \mathbb{N}$ and $s > 1$,

$$(7.8) \quad \begin{aligned} \mathfrak{T}(p, s, q; -1, i) &= 2(-1)^p \sum_{k=0}^{[p/2]} \phi(2k) \binom{p+q-1-2k}{q-1} \phi(s+p+q-2k; 1/4) \\ &\quad + 2(-1)^p \sum_{k=0}^{[q/2]} \phi(2k) \binom{p+q-1-2k}{p-1} \phi(s+p+q-2k; -1/4), \end{aligned}$$

$$(7.9) \quad \begin{aligned} \mathfrak{T}(p, s, q; -i, -1) &= (-1)^p \sum_{l=0}^p \Lambda(l; -i) (-1)^l \binom{p+q-1-l}{q-1} \phi(s+p+q-l) \\ &\quad + (-1)^p \sum_{l=0}^q \Lambda(l; -i) \binom{p+q-1-l}{p-1} \phi(s+p+q-l; -1/4), \end{aligned}$$

$$(7.10) \quad \begin{aligned} \mathfrak{T}(p, s, q; -i, i) &= (-1)^p \sum_{l=0}^p \Lambda(l; -i) (-1)^l \binom{p+q-1-l}{q-1} \phi(s+p+q-l; 1/4) \\ &\quad + (-1)^p \sum_{l=0}^q \Lambda(l; -i) \binom{p+q-1-l}{p-1} \phi(s+p+q-l), \end{aligned}$$

$$(7.11) \quad \begin{aligned} \mathfrak{T}(p, s, q; -1, -i) &= 2(-1)^p \sum_{k=0}^{[p/2]} \phi(2k) \binom{p+q-1-2k}{q-1} \phi(s+q+j; -1/4) \\ &\quad + 2(-1)^p \sum_{k=0}^{[q/2]} \phi(2k) \binom{p+q-1-2k}{p-1} \phi(s+p+q-2k; 1/4), \end{aligned}$$

$$(7.12) \quad \begin{aligned} \mathfrak{T}(p, s, q; i, -1) &= (-1)^p \sum_{l=0}^p \Lambda(l; i) (-1)^l \binom{p+q-1-l}{q-1} \phi(s+p+q-l) \\ &\quad + (-1)^p \sum_{l=0}^q \Lambda(l; i) \binom{p+q-1-l}{p-1} \phi(s+p+q-l; 1/4), \end{aligned}$$

(7.13)

$$\begin{aligned} \mathfrak{T}(p, s, q; i, -i) &= (-1)^p \sum_{l=0}^p \Lambda(l; i) (-1)^l \binom{p+q-1-l}{q-1} \phi(s+p+q-l; -1/4) \\ &\quad + (-1)^p \sum_{l=0}^q \Lambda(l; i) \binom{p+q-1-l}{p-1} \phi(s+p+q-l). \end{aligned}$$

Setting $(x, y, \theta) = (-i, i, \pi/2)$ and $(i, -i, -\pi/2)$ in (5.2), and using Lemma 15, we obtain the following.

Theorem 17. For $p, q, a, b, c \in \mathbb{N}$,

(7.14)

$$\begin{aligned} &\zeta_3((p, q, s, a, b, c), \lambda_1^\vee; SU(4)) + (-1)^p \zeta_3((p, a, s, q, c, b), \lambda_1^\vee; SU(4)) \\ &+ (-1)^{p+a} \zeta_3((q, a, c, p, s, b), \lambda_1^\vee; SU(4)) + (-1)^{p+a+c} \zeta_3((q, s, c, b, a, p), \lambda_1^\vee; SU(4)) \\ &+ (-1)^q \zeta_3((a, q, b, p, s, c), \lambda_1^\vee; SU(4)) + (-1)^{q+b} \zeta_3((a, s, b, c, q, p), \lambda_1^\vee; SU(4)) \\ &+ (-1)^{q+a} \zeta_3((a, p, b, q, c, s), \lambda_1^\vee; SU(4)) + (-1)^{q+a+b} \zeta_3((a, c, b, s, p, q), \lambda_1^\vee; SU(4)) \\ &+ (-1)^{q+a+b+c} \zeta_3((s, c, p, a, b, q), \lambda_1^\vee; SU(4)) + (-1)^{p+q+a} \zeta_3((q, p, c, a, b, s), \lambda_1^\vee; SU(4)) \\ &+ (-1)^{p+q+a+c} \zeta_3((q, b, c, s, p, a), \lambda_1^\vee; SU(4)) \\ &+ (-1)^{p+q+a+b+c} \zeta_3((s, b, p, q, c, a), \lambda_1^\vee; SU(4)) \\ &= (-1)^p \sum_{j=0}^p \Lambda(j; -i) (-1)^j \sum_{\omega=0}^{p-j} \binom{\omega+a-1}{\omega} \binom{p+c-j-\omega-1}{c-1} \\ &\quad \times \mathfrak{T}(q+a+\omega, s, p+b+c-j-\omega; -1, i) \\ &+ (-1)^{p+a} \sum_{j=0}^c \Lambda(j; -i) \sum_{\omega=0}^{c-j} \binom{\omega+a-1}{\omega} \binom{p+c-j-\omega-1}{p-1} \\ &\quad \times \mathfrak{T}(q, s+a+\omega, p+b+c-j-\omega; -i, -1) \\ &+ (-1)^p \sum_{j=0}^a \Lambda(j; -i) \sum_{\omega=0}^{p-1} \binom{\omega+a-j}{\omega} \binom{p+c-2-\omega}{c-1} \\ &\quad \times \mathfrak{T}(q+a-j+\omega+1, s, p+b+c-1-\omega; -i, i) \\ &+ (-1)^{p+a} \sum_{j=0}^a \Lambda(j; -i) (-1)^j \sum_{\omega=0}^{c-1} \binom{\omega+a-j}{\omega} \binom{p+c-2-\omega}{p-1} \\ &\quad \times \mathfrak{T}(q, s+a-j+\omega+1, p+b+c-1-\omega; -i, i) \end{aligned}$$

and

(7.15)

$$\begin{aligned} &\zeta_3((p, q, s, a, b, c), \lambda_3^\vee; SU(4)) + (-1)^p \zeta_3((p, a, s, q, c, b), \lambda_3^\vee; SU(4)) \\ &+ (-1)^{p+a} \zeta_3((q, a, c, p, s, b), \lambda_3^\vee; SU(4)) + (-1)^{p+a+c} \zeta_3((q, s, c, b, a, p), \lambda_3^\vee; SU(4)) \end{aligned}$$

$$\begin{aligned}
& + (-1)^q \zeta_3((a, q, b, p, s, c), \lambda_3^\vee; SU(4)) + (-1)^{q+b} \zeta_3((a, s, b, c, q, p), \lambda_3^\vee; SU(4)) \\
& + (-1)^{q+a} \zeta_3((a, p, b, q, c, s), \lambda_3^\vee; SU(4)) + (-1)^{q+a+b} \zeta_3((a, c, b, s, p, q), \lambda_3^\vee; SU(4)) \\
& + (-1)^{q+a+b+c} \zeta_3((s, c, p, a, b, q), \lambda_3^\vee; SU(4)) + (-1)^{p+q+a} \zeta_3((q, p, c, a, b, s), \lambda_3^\vee; SU(4)) \\
& + (-1)^{p+q+a+c} \zeta_3((q, b, c, s, p, a), \lambda_3^\vee; SU(4)) \\
& + (-1)^{p+q+a+b+c} \zeta_3((s, b, p, q, c, a), \lambda_3^\vee; SU(4)) \\
& = (-1)^p \sum_{j=0}^p \Lambda(j; i) (-1)^j \sum_{\omega=0}^{p-j} \binom{\omega+a-1}{\omega} \binom{p+c-j-\omega-1}{c-1} \\
& \quad \times \mathfrak{T}(q+a+\omega, s, p+b+c-j-\omega; -1, -i) \\
& + (-1)^{p+a} \sum_{j=0}^c \Lambda(j; i) \sum_{\omega=0}^{c-j} \binom{\omega+a-1}{\omega} \binom{p+c-j-\omega-1}{p-1} \\
& \quad \times \mathfrak{T}(q, s+a+\omega, p+b+c-j-\omega; i, -1) \\
& + (-1)^p \sum_{j=0}^a \Lambda(j; i) \sum_{\omega=0}^{p-1} \binom{\omega+a-j}{\omega} \binom{p+c-2-\omega}{c-1} \\
& \quad \times \mathfrak{T}(q+a-j+\omega+1, s, p+b+c-1-\omega; i, -i) \\
& + (-1)^{p+a} \sum_{j=0}^a \Lambda(j; i) (-1)^j \sum_{\omega=0}^{c-1} \binom{\omega+a-j}{\omega} \binom{p+c-2-\omega}{p-1} \\
& \quad \times \mathfrak{T}(q, s+a-j+\omega+1, p+b+c-1-\omega; i, -i)
\end{aligned}$$

hold for $s \in \mathbb{C}$ except for singularities of functions on both sides. Moreover, since $\Lambda(j; \pm i)$ and $\mathfrak{T}(p, s, q; x, y)$ satisfy (7.3)-(7.4) and (7.8)-(7.13), respectively, we find that the right-hand sides of (7.14) and (7.15) can be written in terms of $\zeta(s)$ and $L(s, \chi_4)$.

Setting $(a, b, c, p, q, s) = (2k, 2k, 2k, 2k, 2k, 2k)$ for $k \in \mathbb{N}$, we obtain

$$\zeta_3((2k, 2k, 2k, 2k, 2k, 2k), \lambda_j^\vee; SU(4)) \in \mathbb{Q} \cdot \pi^{12k} \quad (j = 1, 3),$$

because, since χ_4 is an odd character, $L(2l+1, \chi_4) \in \mathbb{Q} \cdot \pi^{2l+1}$ (see, for example, [5, p.12]). This is again an example of [17, Theorem 3.2].

Now we note that

$$(7.16) \quad 1 - i^{3l+2m+n} + (-1)^{l+n} - i^{l+2m+3n} = \begin{cases} 4 & \text{if } l+2m+3n \equiv 2 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$$

This is because $(-1)^{l+n} = 1$ and $i^{3l+2m+n} = i^{l+2m+3n}$ when l and n are both even or both odd, while $(-1)^{l+n} = -1$ and $i^{3l+2m+n} = -i^{l+2m+3n}$ otherwise. Therefore

$$\begin{aligned}
(7.17) \quad & \zeta_3(\mathbf{s}, \{\mathbf{0}\}; PU(4)) \\
& = \frac{1}{4} \left(\zeta_3(\mathbf{s}, A_3) - \zeta_3(\mathbf{s}, \lambda_1^\vee; SU(4)) + \zeta_3(\mathbf{s}, \lambda_2^\vee; SU(4)) - \zeta_3(\mathbf{s}, \lambda_3^\vee; SU(4)) \right),
\end{aligned}$$

which is further equal to

$$\frac{1}{4} \left(2\zeta_3(\mathbf{s}, SO(6)) - \zeta_3(\mathbf{s}, \lambda_1^\vee; SU(4)) - \zeta_3(\mathbf{s}, \lambda_3^\vee; SU(4)) \right)$$

by (6.1). Hence it follows from (3.13) and (3.14) that

(7.18)

$$\zeta_3((s_1, s_2, s_3, s_4, s_5, s_6), \{\mathbf{0}\}; PU(4)) = \zeta_3((s_3, s_2, s_1, s_5, s_4, s_6), \{\mathbf{0}\}; PU(4)).$$

Using (7.17) and combining Theorem 12 and Theorem 17, we obtain the following functional relation among $\zeta_3(\mathbf{s}, \{\mathbf{0}\}; PU(4))$, $\zeta(s)$ and $L(s, \chi_4)$.

Theorem 18. For $p, q, a, b, c \in \mathbb{N}$,

(7.19)

$$\begin{aligned} & \zeta_3((p, q, s, a, b, c), \mathbf{0}; PU(4)) + (-1)^p \zeta_3((p, a, s, q, c, b), \mathbf{0}; PU(4)) \\ & + (-1)^{p+a} \zeta_3((q, a, c, p, s, b), \mathbf{0}; PU(4)) + (-1)^{p+a+c} \zeta_3((q, s, c, b, a, p), \mathbf{0}; PU(4)) \\ & + (-1)^q \zeta_3((a, q, b, p, s, c), \mathbf{0}; PU(4)) + (-1)^{q+b} \zeta_3((a, s, b, c, q, p), \mathbf{0}; PU(4)) \\ & + (-1)^{q+a} \zeta_3((a, p, b, q, c, s), \mathbf{0}; PU(4)) + (-1)^{q+a+b} \zeta_3((a, c, b, s, p, q), \mathbf{0}; PU(4)) \\ & + (-1)^{q+a+b+c} \zeta_3((s, c, p, a, b, q), \mathbf{0}; PU(4)) + (-1)^{p+q+a} \zeta_3((q, p, c, a, b, s), \mathbf{0}; PU(4)) \\ & + (-1)^{p+q+a+c} \zeta_3((q, b, c, s, p, a), \mathbf{0}; PU(4)) \\ & + (-1)^{p+q+a+b+c} \zeta_3((s, b, p, q, c, a), \mathbf{0}; PU(4)) \\ & = \frac{1}{4} (J_0 - J_1 + J_2 - J_3), \end{aligned}$$

where J_0, J_1, J_2, J_3 are the right-hand sides of (5.4), (7.14), (5.3), (7.15), respectively.

Setting $(a, b, c, p, q, s) = (2k, 2k, 2k, 2k, 2k, 2k)$ for $k \in \mathbb{N}$, we obtain

$$\zeta_3((2k, 2k, 2k, 2k, 2k, 2k), \mathbf{0}; PU(4)) \in \mathbb{Q} \cdot \pi^{12k}.$$

Also, set $(a, b, c, p, q, s) = (2k+1, 2k+1, 2k+1, 2k+1, 2k, 2k+1)$ in (7.14), (7.15) and (7.19). Then, by the same method as in the proof of Proposition 10, we obtain the following.

Proposition 19. For $k \in \mathbb{N}$ and $j = 1, 3$,

$$\begin{aligned} & \zeta_3((2k, 2k+1, 2k+1, 2k+1, 2k+1, 2k+1), \lambda_j^\vee; SU(4)) \\ & \in \mathbb{Q}[\{\zeta(j+1), L(j, \chi_4) \mid j \in \mathbb{N}\}], \end{aligned}$$

and so

$$\begin{aligned} & \zeta_3((2k, 2k+1, 2k+1, 2k+1, 2k+1, 2k+1), \{\mathbf{0}\}; PU(4)) \\ & \in \mathbb{Q}[\{\zeta(j+1), L(j; \chi_4) \mid j \in \mathbb{N}\}], \end{aligned}$$

where the rational coefficients can be determined explicitly.

Example 20. Setting $(a, b, c, p, q, s) = (2, 2, 2, 2, 2, s)$ in (7.14) and (7.15), and using the data $L(1, \chi_4) = \pi/4$, $L(3, \chi_4) = \pi^3/32$, $L(5, \chi_4) = (5/1536)\pi^5$, we obtain

$$\begin{aligned}
& 2 \left\{ \zeta_3((s, 2, 2, 2, 2, 2), \lambda_1^\vee; SU(4)) + \zeta_3((2, s, 2, 2, 2, 2), \lambda_1^\vee; SU(4)) \right. \\
& \quad + \zeta_3((2, 2, s, 2, 2, 2), \lambda_1^\vee; SU(4)) + \zeta_3((2, 2, 2, s, 2, 2), \lambda_1^\vee; SU(4)) \\
& \quad \left. + \zeta_3((2, 2, 2, 2, s, 2), \lambda_1^\vee; SU(4)) + \zeta_3((2, 2, 2, 2, 2, s), \lambda_1^\vee; SU(4)) \right\} \\
& = (372 \cdot 2^{-s-10} + 306) (2^{-s-9} - 1) \zeta(s+10) + 100\pi L(s+9, \chi_4) \\
& \quad + \left(7 \cdot 2^{-s-8} + \frac{32}{3} \right) (2^{-s-7} - 1) \pi^2 \zeta(s+8) + \frac{17}{6} \pi^3 L(s+7, \chi_4) \\
& \quad + \left(\frac{113 \cdot 2^{-s-6}}{1440} + \frac{1}{288} \right) (2^{-s-5} - 1) \pi^4 \zeta(s+6) \\
& \quad + \frac{1}{32} \pi^5 L(s+5, \chi_4) + \frac{289 \cdot 2^{-s-4}}{241920} (2^{-s-3} - 1) \pi^6 \zeta(s+4),
\end{aligned}$$

and

$$\begin{aligned}
& 2 \left\{ \zeta_3((s, 2, 2, 2, 2, 2), \lambda_3^\vee; SU(4)) + \zeta_3((2, s, 2, 2, 2, 2), \lambda_3^\vee; SU(4)) \right. \\
& \quad + \zeta_3((2, 2, s, 2, 2, 2), \lambda_3^\vee; SU(4)) + \zeta_3((2, 2, 2, s, 2, 2), \lambda_3^\vee; SU(4)) \\
& \quad \left. + \zeta_3((2, 2, 2, 2, s, 2), \lambda_3^\vee; SU(4)) + \zeta_3((2, 2, 2, 2, 2, s), \lambda_3^\vee; SU(4)) \right\} \\
& = (372 \cdot 2^{-s-10} + 306) (2^{-s-9} - 1) \zeta(s+10) + 100\pi L(s+9, \chi_4) \\
& \quad + \left(7 \cdot 2^{-s-8} + \frac{32}{3} \right) (2^{-s-7} - 1) \pi^2 \zeta(s+8) + \frac{17}{6} \pi^3 L(s+7, \chi_4) \\
& \quad + \left(\frac{113 \cdot 2^{-s-6}}{1440} + \frac{1}{288} \right) (2^{-s-5} - 1) \pi^4 \zeta(s+6) \\
& \quad + \frac{1}{32} \pi^5 L(s+5, \chi_4) + \frac{289 \cdot 2^{-s-4}}{241920} (2^{-s-3} - 1) \pi^6 \zeta(s+4).
\end{aligned}$$

Note that the reason why the right-hand sides of the above two formulas are the same is given by (3.13). Setting $(a, b, c, p, q, s) = (2, 2, 2, 2, 2, s)$ in (7.19), we obtain

$$\begin{aligned}
& 2 \left\{ \zeta_3((s, 2, 2, 2, 2, 2), \mathbf{0}; PU(4)) + \zeta_3((2, s, 2, 2, 2, 2), \mathbf{0}; PU(4)) \right. \\
& \quad + \zeta_3((2, 2, s, 2, 2, 2), \mathbf{0}; PU(4)) + \zeta_3((2, 2, 2, s, 2, 2), \mathbf{0}; PU(4)) \\
& \quad \left. + \zeta_3((2, 2, 2, 2, s, 2), \mathbf{0}; PU(4)) + \zeta_3((2, 2, 2, 2, 2, s), \mathbf{0}; PU(4)) \right\} \\
& = \left(-\frac{93 \cdot 2^{-2s}}{262144} + \frac{33 \cdot 2^{-s}}{512} + 306 \right) \zeta(s+10) - 50\pi L(s+9, \chi_4) \\
& \quad + \left(-\frac{7 \cdot 2^{-2s}}{65536} - \frac{19 \cdot 2^{-s}}{1536} - \frac{49}{3} \right) \pi^2 \zeta(s+8) - \frac{17}{12} \pi^3 L(s+7, \chi_4)
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{113 \cdot 2^{-2s}}{5898240} - \frac{323 \cdot 2^{-s}}{61440} + \frac{353}{576} \right) \pi^4 \zeta(s+6) - \frac{\pi^5}{64} L(s+5, \chi_4) \\
& + \left(-\frac{289 \cdot 2^{-2s}}{61931520} - \frac{31 \cdot 2^{-s}}{7741440} + \frac{1}{720} \right) \pi^6 \zeta(s+4).
\end{aligned}$$

In particular, setting $s = 2$, we again obtain (3.18).

Example 21. Similarly to Example 11, We obtain, for example,

$$\begin{aligned}
& \zeta_3((2, 3, 3, 3, 3, 3), \lambda_1^\vee; SU(4)) = \zeta_3((2, 3, 3, 3, 3, 3), \lambda_3^\vee; SU(4)) \\
& = \frac{2125\pi^8}{45097156608} \zeta(9) + \frac{11\pi^7}{15360} L(10, \chi_4) - \frac{440049247\pi^6}{225485783040} \zeta(11) \\
& + \frac{13\pi^5}{96} L(12, \chi_4) - \frac{1056786549\pi^4}{2147483648} \zeta(13) + 11\pi^3 L(14, \chi_4) \\
& - \frac{199887481225\pi^2}{4294967296} \zeta(15) + 399\pi L(16, \chi_4) - \frac{2424501730875}{2147483648} \zeta(17),
\end{aligned}$$

and

$$\begin{aligned}
& \zeta_3((2, 3, 3, 3, 3, 3), \{\mathbf{0}\}; PU(4)) \\
& = \frac{1111987\pi^8}{90194313216} \zeta(9) - \frac{11\pi^7}{30720} L(10, \chi_4) + \frac{11180759837\pi^6}{1352914698240} \zeta(11) \\
& - \frac{13\pi^5}{192} L(12, \chi_4) + \frac{170862984923\pi^4}{64424509440} \zeta(13) - \frac{11\pi^3}{2} L(14, \chi_4) \\
& - \frac{1504872383333\pi^2}{25769803776} \zeta(15) - \frac{399\pi}{2} L(16, \chi_4) + \frac{4849040457275}{4294967296} \zeta(17).
\end{aligned}$$

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